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Coherent Exclusive Exponentiation CEEX: the Case of the Resonant e^+e^- Collision

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Abstract

We present the first-order coherent exclusive exponentiation (CEEX) scheme, with the full control over spin polarization for all fermions. In particular it is applicable to difficult case of narrow resonances. The resulting spin amplitudes and the differential distributions are given in a form ready for their implementation in the Monte Carlo event generator. The initial-final state interferences are under control. The way is open to the use of the exact amplitudes for two and more hard photons, using Weyl-spinor techniques, without giving up the advantages of the exclusive exponentiation, of the Yennie-Frautschi-Suura type.

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1 Introduction: What is the problem?

The problem addressed in this work is: How to describe, consistently in the process $e^+e^- \to f\bar{f}$, the coherent emission of *initial* state radiation (ISR) and *final* state radiation (FSR) of *soft* and *hard* photons, providing for cancellations of infrared (IR) divergences from real and virtual photon emission to infinite perturbative order (exponentiation), at the level of completely *exclusive* multiphoton differential distributions, i.e. in the form suitable for implementation in Monte Carlo (MC) event generators? In addition we are looking for the solution that is friendliest to narrow s-channel resonances.

This work is firmly rooted in the work of Yennie, Frautschi and Suura (YFS) on QED exponentiation [1] and its further developments in refs. [2–4]. The present work definitely goes beyond the scope of these previous papers – the main difference is the consequent use of spin amplitudes in the exponentiation. Our work is close in spirit, although not in technical details, to seminal papers of Greco et al. [5,6] on QED exponentiation for narrow resonances. However, it should be stressed that, contrary to refs. [5,6], all our differential multiphoton distributions are completely exclusive (important for MC implementation) and we do include hard photons completely and systematically. In this context, the work of ref. [7] should also be mentioned. It implements QED interferences among e^+ and e^- fermion lines, the analog of the ISR–FSR interferences, for the first time in the exclusive exponentiation. It does not, however, use spin amplitudes for exponentiation as consequently as does the present work; it is also rather strongly limited to exact first order exponentiation in the YFS framework. It is sort of half-way between the present work and the older ones of refs. [2,3]. At the technical level, the methods used here for the construction of the spin amplitudes are essentially those[1] of Kleiss and Stirling (KS) [8,9], with the important supplement of ref. [10], providing for total control of complex phases and/or fermion spin quantization frames. The MC implementation of the present work will soon be available [11] and it will replace two MC programs: KORALB [12], where fermion spin polarizations are implemented exactly, but there is no exponentiation, and KORALZ [13], where exponentiation is included, but the treatment of spin effects is simplified[2].

The present work is essential for any present experiments in e^+e^- colliders and future e^+e^- and $\mu^+\mu^-$ colliders, where the most important new features for data analysis will be inclusion of ISR–FSR interferences and (in the next step) the exact matrix element for emission of 2 and 3 hard photons, in the presence of many additional soft ones.

2 Basic KS/GPS spinors and photon polarizations

The arbitrary massless spinor $u_\lambda(p)$ of momentum p and chirality λ is defined according to KS methods [8,9]. In the following we follow closely the notation of ref. [10] (in particular

[1] We have evaluated several techniques based on Weyl-spinor techniques and we concluded that the technique of KS is best suited for our needs (exponentiation).

[2] The ISR–FSR interference is also neglected in KORALZ, in the main mode with the exponentiation switched on.

we also use $\zeta = \zeta_\downarrow$). In the above framework every spinor is transformed out of the two *constant basic* spinors $\mathfrak{u}_\lambda(\zeta)$, of opposite chirality $\lambda = \pm$, as follows

$$u_\lambda(p) = \frac{1}{\sqrt{2p\cdot\zeta}}\; \not{p}\,\mathfrak{u}_{-\lambda}(\zeta), \quad \mathfrak{u}_+(\zeta) = \not{\eta}\,\mathfrak{u}_-(\zeta), \quad \eta^2 = -1, \quad (\eta\zeta) = 0. \tag{1}$$

The usual relations hold: $\not{\zeta}\,\mathfrak{u}_\lambda(\zeta) = 0$, $\omega_\lambda \mathfrak{u}_\lambda(\zeta) = \mathfrak{u}_\lambda(\zeta)$, $\mathfrak{u}_\lambda(\zeta)\bar{\mathfrak{u}}_\lambda(\zeta) = \not{\zeta}\,\omega_\lambda$, $\not{p}\,u_\lambda(p) = 0$, $\omega_\lambda u_\lambda(p) = u_\lambda(p)$, $u_\lambda(p)\bar{u}_\lambda(p) = \not{p}\,\omega_\lambda$, where $\omega_\lambda = \frac{1}{2}(1+\lambda\gamma_5)$. Spinors for the massive particle with four-momentum p (with $p^2 = m^2$) and spin projection $\lambda/2$ are defined in terms of massless spinors

$$u(p,\lambda) = u_\lambda(p_\zeta) + \frac{m}{\sqrt{2p\zeta}}\mathfrak{u}_{-\lambda}(\zeta), \qquad v(p,\lambda) = u_{-\lambda}(p_\zeta) - \frac{m}{\sqrt{2p\zeta}}\mathfrak{u}_\lambda(\zeta), \tag{2}$$

where $p_\zeta \equiv \hat{p} \equiv p - \zeta\; m^2/(2\zeta p)$ is the light-cone projection ($p_\zeta^2 = 0$) of the p obtained with the help of the constant auxiliary vector ζ.

The above definition is supplemented in ref. [10] with the precise prescription on spin quantization axes, translation from spin amplitudes to density matrices (also in vector notation) and the methodology of connecting production and decay for unstable fermions. We collectively call these rules global positioning of spin (GPS). Thanks to these we are able to easily introduce polarizations for beams and implement polarization effects for final fermion decays (of τ leptons, t-quarks), for the first time also in the presence of emission of many ISR and FSR photons!

The GPS rules determining spin quantization frame for $u(p,\pm)$ and $v(p,\pm)$ of eq. (2) are summarized as follows: (a) In the rest frame of the fermion, take the z-axis along $-\vec{\zeta}$. (b) Place the x-axis in the plane defined by the z-axis from the previous point and the vector $\vec{\eta}$, in the same half-plane as $\vec{\eta}$. (c) With the y-axis, complete the right-handed system of coordinates. The rest frame defined in this way we call the GPS frame of the particular fermion. See ref. [10] for more details. In the following we shall assume that polarization vectors of beams and of outgoing fermions are defined in their corresponding GPS frames.

The inner product of the two massless spinors is defined as follows

$$s_+(p_1,p_2) \equiv \bar{u}_+(p_1)u_-(p_2), \quad s_-(p_1,p_2) \equiv \bar{u}_-(p_1)u_+(p_2) = -(s_+(p_1,p_2))^*. \tag{3}$$

The above inner product can be evaluated using the Kleiss-Stirling expression

$$s_+(p,q) = 2\;(2p\zeta)^{-1/2}\;(2q\zeta)^{-1/2}\;\left[(p\zeta)(q\eta) - (p\eta)(q\zeta) - i\epsilon_{\mu\nu\rho\sigma}\zeta^\mu\eta^\nu p^\rho q^\sigma\right] \tag{4}$$

in any reference frame. In particular, in the laboratory frame we typically use $\zeta = (1,1,0,0)$ and $\eta = (0,0,1,0)$, which leads to the following "massless" inner product

$$s_+(p,q) = -(q^2 + iq^3)\sqrt{(p^0-p^1)/(q^0-q^1)} + (p^2+ip^3)\sqrt{(q^0-q^1)/(p^0-p^1)}. \tag{5}$$

Equation (2) immediately provides us also with the *inner product* for massive spinors

$$\begin{aligned}
\bar{u}(p_1,\lambda_1)u(p_2,\lambda_2) &= S(p_1,m_1,\lambda_1,p_2,m_2,\lambda_2),\\
\bar{u}(p_1,\lambda_1)v(p_2,\lambda_2) &= S(p_1,m_1,\lambda_1,p_2,-m_2,-\lambda_2),\\
\bar{v}(p_1,\lambda_1)u(p_2,\lambda_2) &= S(p_1,-m_1,-\lambda_1,p_2,m_2,\lambda_2),\\
\bar{v}(p_1,\lambda_1)v(p_2,\lambda_2) &= S(p_1,-m_1,-\lambda_1,p_2,-m_2,-\lambda_2),
\end{aligned} \tag{6}$$

where

$$S(p_1, m_1, \lambda_1, p_2, m_2, \lambda_2) = \delta_{\lambda_1,-\lambda_2} s_{\lambda_1}(p_{1\zeta}, p_{2\zeta}) + \delta_{\lambda_1,\lambda_2} \left(m_1 \sqrt{\frac{2\zeta p_2}{2\zeta p_1}} + m_2 \sqrt{\frac{2\zeta p_1}{2\zeta p_2}} \right). \quad (7)$$

In our spinor algebra we shall exploit the completeness relations

$$\begin{aligned} &\not{p} + m = \sum_\lambda u(p,\lambda)\bar{u}(p,\lambda), \quad \not{p} - m = \sum_\lambda v(p,\lambda)\bar{v}(p,\lambda), \\ &\not{k} = \sum_\lambda u(k,\lambda)\bar{u}(k,\lambda), \quad k^2 = 0. \end{aligned} \quad (8)$$

For a circularly polarized photon with four-momentum k and helicity $\sigma = \pm 1$ we adopt the KS choice (see also ref. [14]) of polarization vector[3]

$$(\epsilon^\mu_\sigma(\beta))^* = \frac{\bar{u}_\sigma(k)\gamma^\mu u_\sigma(\beta)}{\sqrt{2}\,\bar{u}_{-\sigma}(k)u_\sigma(\beta)}, \quad (\epsilon^\mu_\sigma(\zeta))^* = \frac{\bar{u}_\sigma(k)\gamma^\mu \mathfrak{u}_\sigma(\zeta)}{\sqrt{2}\,\bar{u}_{-\sigma}(k)\mathfrak{u}_\sigma(\zeta)}, \quad (9)$$

where β is an arbitrary light-like four-vector $\beta^2 = 0$. The second choice with $\mathfrak{u}_\sigma(\zeta)$ (not exploited in [8]) often leads to simplifications in the resulting photon emission amplitudes. Using the Chisholm identity[4]

$$\bar{u}_\sigma(k)\gamma_\mu u_\sigma(\beta)\,\gamma^\mu = 2u_\sigma(\beta)\,\bar{u}_\sigma(k) + 2u_{-\sigma}(k)\,\bar{u}_{-\sigma}(\beta), \quad (10)$$

$$\bar{u}_\sigma(k)\gamma_\mu \mathfrak{u}_\sigma(\zeta)\,\gamma^\mu = 2\mathfrak{u}_\sigma(\zeta)\,\bar{u}_\sigma(k) - 2u_{-\sigma}(k)\,\bar{\mathfrak{u}}_{-\sigma}(\zeta), \quad (11)$$

we get two useful expressions, equivalent to eq. (9):

$$\begin{aligned} (\not{\epsilon}_\sigma(k,\beta))^* &= \frac{\sqrt{2}}{\bar{u}_{-\sigma}(k)u_\sigma(\beta)} \left[u_\sigma(\beta)\bar{u}_\sigma(k) + u_{-\sigma}(k)\bar{u}_{-\sigma}(\beta)\right] \\ (\not{\epsilon}_\sigma(k,\zeta))^* &= \frac{\sqrt{2}}{\sqrt{2\zeta k}} \left[\mathfrak{u}_\sigma(\zeta)\bar{u}_\sigma(k) - u_{-\sigma}(k)\bar{\mathfrak{u}}_{-\sigma}(\zeta)\right]. \end{aligned} \quad (12)$$

In the evaluation of photon emission spin amplitudes we shall use the following important building block – the elements of the "transition matrices" U and V defined as follows

$$\begin{aligned} \bar{u}(p_1,\lambda_1)\ \not{\epsilon}^*_\sigma(k,\beta)\ u(p_2,\lambda_2) &= U \binom{k}{\sigma} \begin{bmatrix} p_1 & p_2 \\ \lambda_1 & \lambda_2 \end{bmatrix} = U^\sigma_{\lambda_1,\lambda_2}(k,p_1,m_1,p_2,m_2), \\ \bar{v}(p_1,\lambda_1)\ \not{\epsilon}^*_\sigma(k,\zeta)\ v(p_2,\lambda_2) &= V \binom{k}{\sigma} \begin{bmatrix} p_1 & p_2 \\ \lambda_1 & \lambda_2 \end{bmatrix} = V^\sigma_{\lambda_1,\lambda_2}(k,p_1,m_1,p_2,m_2). \end{aligned} \quad (13)$$

[3] Contrary to other papers on Weyl spinor techniques [8, 15] we keep here the explicitly complex conjugation in ϵ. This conjugation is cancelled by another conjugation following from Feynman rules, but only for outgoing photons, not for beam photon, as in the Compton process, see ref. [16].

[4] For $\beta = \zeta$ the identity is slightly different because of the additional minus sign in the "line-reversal" rule, i.e. $\bar{u}_\sigma(k)\gamma^\mu \mathfrak{u}_\sigma(\zeta) = -\bar{\mathfrak{u}}_{-\sigma}(\zeta)\gamma^\mu u_{-\sigma}(k)$, in contrast to the usual $\bar{u}_\sigma(k)\gamma^\mu u_\sigma(\beta) = +\bar{u}_{-\sigma}(\beta)\gamma^\mu u_{-\sigma}(k)$.

In the case of $\mathfrak{u}_\sigma(\zeta)$ the above transition matrices are rather simple[5]:

$$U^+(k,p_1,m_1,p_2,m_2) = \sqrt{2}\begin{bmatrix} \sqrt{\frac{2\zeta p_2}{2\zeta k}}s_+(k,\hat{p}_1), & 0 \\ m_2\sqrt{\frac{2\zeta p_1}{2\zeta p_2}} - m_1\sqrt{\frac{2\zeta p_2}{2\zeta p_1}}, & \sqrt{\frac{2\zeta p_1}{2\zeta k}}s_+(k,\hat{p}_2) \end{bmatrix}, \tag{14}$$

$$U^-_{\lambda_1,\lambda_2}(k,p_1,m_1,p_2,m_2) = \left[-U^+_{\lambda_2,\lambda_1}(k,p_2,m_2,p_1,m_1)\right]^*, \tag{15}$$

$$V^\sigma_{\lambda_1,\lambda_2}(k,p_1,m_1,p_2,m_2) = U^\sigma_{-\lambda_1,-\lambda_2}(k,p_1,-m_1,p_2,-m_2). \tag{16}$$

The more general case with $u_\sigma(\beta)$ looks a little bit more complicated:

$$U^+(k,p_1,m_1,p_2,m_2) = \frac{\sqrt{2}}{s_-(k,\beta)} \times$$
$$\begin{bmatrix} s_+(\hat{p}_1,k)s_-(\beta,\hat{p}_2) + m_1 m_2\sqrt{\frac{2\zeta\beta}{2\zeta p_1}\frac{2\zeta k}{2\zeta p_2}}, & m_1\sqrt{\frac{2\zeta\beta}{2\zeta p_1}}s_+(k,\hat{p}_2) + m_2\sqrt{\frac{2\zeta\beta}{2\zeta p_2}}s_+(\hat{p}_1,k) \\ m_1\sqrt{\frac{2\zeta k}{2\zeta p_1}}s_-(\beta,\hat{p}_2) + m_2\sqrt{\frac{2\zeta k}{2\zeta p_2}}s_-(\hat{p}_1,\beta), & s_-(\hat{p}_1,\beta)s_+(k,\hat{p}_2) + m_1 m_2\sqrt{\frac{2\zeta\beta}{2\zeta p_1}\frac{2\zeta k}{2\zeta p_2}} \end{bmatrix}, \tag{17}$$

with the same relations (15) and (16). In the above the following numbering of elements in matrices U and V is adopted

$$\{(\lambda_1,\lambda_2)\} = \begin{bmatrix} (++) & (+-) \\ (-+) & (--) \end{bmatrix}. \tag{18}$$

When analysing the soft real photon limit we shall exploit the following important *diagonality* property[6]

$$U\binom{k}{\sigma}\begin{bmatrix} p & p \\ \lambda_1 & \lambda_2 \end{bmatrix} = V\binom{k}{\sigma}\begin{bmatrix} p & p \\ \lambda_1 & \lambda_2 \end{bmatrix} = b_\sigma(k,p)\;\delta_{\lambda_1\lambda_2}, \tag{19}$$

$$b_\sigma(k,p) = \sqrt{2}\;\frac{\bar{u}_\sigma(k)\;\not{p}\;\mathfrak{u}_\sigma(\zeta)}{\bar{u}_{-\sigma}(k)\mathfrak{u}_\sigma(\zeta)} = \sqrt{2}\;\sqrt{\frac{2\zeta p}{2\zeta k}}s_\sigma(k,\hat{p}), \tag{20}$$

which also holds in the general case of $u_\sigma(\beta)$, where

$$b_\sigma(k,p) = \frac{\sqrt{2}}{s_{-\sigma}(k,\beta)}\left(s_{-\sigma}(\beta,\hat{p})s_\sigma(\hat{p},k) + \frac{m^2}{2\zeta\hat{p}}\;\sqrt{(2\beta\zeta)\;(2\zeta k)}\right). \tag{21}$$

3 Born spin amplitudes

Let us calculate lowest order spin amplitudes for $e^-(p_1)e^+(p_2) \to f(p_3)\bar{f}(p_4)$. For the moment we require $f \neq e$. Using our basic massive spinors of eq. (2) with definite GPS

[5] Our U and V matrices are not the same as the M-matrices of ref. [9], but rather products of several of those.

[6] Let us also keep in mind the relation $b_{-\sigma}(k,p) = -(b_\sigma(k,p))^*$, which can save time in the numerical calculations.

helicities and Feynman rules, we define

$$\begin{aligned}
&\mathfrak{B}\left[{}^{p}_{\lambda}\right](X) = \mathfrak{B}\left[{}^{p_1\,p_2\,p_3\,p_4}_{\lambda_1\lambda_2\lambda_3\lambda_4}\right](X) = ie^2 \sum_{B=\gamma,Z} \frac{\bar{v}(p_2,\lambda_2)\gamma^\mu G^{e,B} u(p_1,\lambda_1)\ \ \bar{u}(p_3,\lambda_3)\gamma_\mu G^{f,B} v(p_4,\lambda_4)}{X^2 - {M_B}^2 + i\Gamma_B X^2/M_B},\\
&\qquad G^{e,B} = \sum_{\lambda=\pm} \omega_\lambda g^{e,B}_\lambda, \quad G^{f,B} = \sum_{\lambda=\pm} \omega_\lambda g^{f,B}_\lambda,
\end{aligned} \tag{22}$$

where $g^{f,B}_\lambda$ are the usual chiral ($\lambda = +1,-1 = R,L$) coupling constants of the vector boson $B = \gamma, Z$ to fermion f in units of the elementary charge e.

Spinor products are reorganized with the help of the Chisholm identity (10), which applies assuming that electron spinors are massless, and the inner product of eq. (7):

$$\mathfrak{B}\left[{}^{p}_{\lambda}\right](X) = 2ie^2 \sum_{B=\gamma,Z} \frac{\delta_{\lambda_1,-\lambda_2}\left[\, g^{e,B}_{\lambda_1} g^{f,B}_{-\lambda_1}\, T_{\lambda_3\lambda_1}\, T'_{\lambda_2\lambda_4} + g^{e,B}_{\lambda_1} g^{f,B}_{\lambda_1}\, U'_{\lambda_3\lambda_2}\, U_{\lambda_1\lambda_4}\right]}{X^2 - {M_B}^2 + i\Gamma_B X^2/M_B}, \tag{23}$$

where

$$\begin{aligned}
T_{\lambda_3\lambda_1} &= \bar{u}(p_3,\lambda_3)u(p_1,\lambda_1) = S(p_3,m_3,\lambda_3,p_1,0,\lambda_1),\\
T'_{\lambda_2\lambda_4} &= \bar{v}(p_2,\lambda_2)v(p_4,\lambda_4) = S(p_2,0,-\lambda_2,p_4,-m_4,-\lambda_4),\\
U'_{\lambda_3\lambda_2} &= \bar{u}(p_3,\lambda_3)v(p_2,-\lambda_2) = S(p_3,m_3,\lambda_3,p_2,0,\lambda_2),\\
U_{\lambda_1\lambda_4} &= \bar{u}(p_1,-\lambda_1)v(p_4,\lambda_4) = S(p_1,0,-\lambda_1,p_4,-m_4,-\lambda_4).
\end{aligned} \tag{24}$$

We understand that the total s-channel four-momentum X is always *the* four-vector that enters the s-channel vector boson propagators. Let us stress that the above Born spin amplitudes will be used for p_i, which *do not necessarily obey* the four-momentum conservation $p_1+p_2 = p_3+p_4$. This is necessary because, in the presence of the bremsstrahlung photons, the relation $X = p_1+p_2 = p_3+p_4$ does not usually hold. Furthermore, any of the p_i may, and occasionally will, be replaced by the momentum k of one of photons. In this case, the spinor into which k enters as an argument is understood to be massless.

4 First order, one virtual photon

The $\mathcal{O}(\alpha^1)$ contribution with one virtual and zero real photon reads

$$\mathcal{M}^{(1)}_0\left[{}^{p}_{\lambda}\right](X) = \mathfrak{B}\left[{}^{p}_{\lambda}\right](X)\left[1 + Q_e^2 F_1(s,m_\gamma) + Q_f^2 F_1(s,m_\gamma)\right] + \mathcal{M}_{\rm box}\left[{}^{p}_{\lambda}\right](X), \tag{25}$$

where F_1 is the standard electric form-factor regularized with photon mass. We omit, for the moment, the magnetic form-factor F_2; this is justified for light final fermions. It will be restored in the future. In F_1 we keep the exact final fermion mass.

In the present work we use spin amplitudes for γ-γ and γ-Z boxes in the small mass approximation $m_e^2/s \to 0, m_f^2/s \to 0$, following refs. [17, 18],

$$\begin{aligned}
\mathcal{M}_{\rm Box}\left[{}^{p}_{\lambda}\right](X) = 2ie^2 \sum_{B=\gamma,Z} &\frac{g^{e,B}_{\lambda_1} g^{f,B}_{-\lambda_1}\, T_{\lambda_3\lambda_1} T'_{\lambda_2\lambda_4} + g^{e,B}_{\lambda_1} g^{f,B}_{\lambda_1}\, U'_{\lambda_3\lambda_2} U_{\lambda_1\lambda_4}}{X^2 - {M_B}^2 + i\Gamma_B X^2/M_B}\ \delta_{\lambda_1,-\lambda_2}\delta_{\lambda_3,-\lambda_4}\\
&\frac{\alpha}{\pi} Q_e Q_f \left[\delta_{\lambda_1,\lambda_3}\ f_{\rm BDP}(\bar{M}_B^2, m_\gamma, s,t,u) - \delta_{\lambda_1,-\lambda_3}\ f_{\rm BDP}(\bar{M}_B^2, m_\gamma, s,u,t)\right],
\end{aligned} \tag{26}$$

where $\bar{M}_Z^2 = M_Z^2 - iM_Z\Gamma_Z$, $\bar{M}_\gamma^2 = m_\gamma^2$, and the function $f_{\rm BDP}$ is defined in eq. (11) of ref. [18]. The Mandelstam variables s, t and u are defined as usual. Since in the rest of our calculation we do not use $m_f^2/s \to 0$, we therefore intend to replace the above box spin amplitudes with the finite-mass results. (NB: For the γ-γ box the spin amplitudes with the exact final fermion mass[7] were given in ref. [12].)

5 First order 1-photon, ISR alone

In order to introduce the notation gradually, let us first consider the 1-photon emission matrix element separately for ISR. The first order, 1-photon, ISR matrix element from the Feynman rule reads

$$\begin{split}\mathcal{M}_1^{\rm ISR}\left(\begin{smallmatrix} p_1\, p_2\, k\\ \lambda_1\,\lambda_2\,\sigma\end{smallmatrix}\right) =& \frac{eQ_e}{2kp_1}\, \bar{v}(p_2,\lambda_2)\, \mathbf{M}_1\, (\not{p}_1 + m - \not{k})\, \not{\epsilon}^\star_\sigma(k)\, u(p_1,\lambda_1)\\ &+\frac{eQ_e}{2kp_2}\, \bar{v}(p_2,\lambda_2)\, \not{\epsilon}^\star_\sigma(k)\, (-\not{p}_2 + m + \not{k})\, \mathbf{M}_1\, u(p_1,\lambda_1),\end{split} \tag{27}$$

where $\mathbf{M_1}$ is the annihilation scattering spinor matrix (including final state spinors). The above expression we split into soft IR parts[8] proportional to $(\not{p} \pm m)$ and non-IR parts proportional to $\not{k}$. Employing the completeness relations of eq. (13) to those parts we obtain:

$$\begin{split}\mathcal{M}_1^{\rm ISR}\left(\begin{smallmatrix} p_1\, p_2\, k\\ \lambda_1\,\lambda_2\,\sigma\end{smallmatrix}\right) =& \frac{eQ_e}{2kp_1} \sum_\rho \mathfrak{B}_1 \left[\begin{smallmatrix} p_1\, p_2\\ \rho\ \lambda_2\end{smallmatrix}\right] U \left(\begin{smallmatrix} k\\ \sigma\end{smallmatrix}\right) \left[\begin{smallmatrix} p_1\, p_1\\ \rho\ \lambda_1\end{smallmatrix}\right] - \frac{eQ_e}{2kp_2} \sum_\rho V \left(\begin{smallmatrix} k\\ \sigma\end{smallmatrix}\right) \left[\begin{smallmatrix} p_2\, p_2\\ \lambda_2\,\rho\end{smallmatrix}\right] \mathfrak{B}_1 \left[\begin{smallmatrix} p_1\, p_2\\ \lambda_1\,\rho\end{smallmatrix}\right]\\ &-\frac{eQ_e}{2kp_1} \sum_\rho \mathfrak{B}_1 \left[\begin{smallmatrix} k p_2\\ \rho \lambda_2\end{smallmatrix}\right] U \left(\begin{smallmatrix} k\\ \sigma\end{smallmatrix}\right) \left[\begin{smallmatrix} k p_1\\ \rho \lambda_1\end{smallmatrix}\right] + \frac{eQ_e}{2kp_2} \sum_\rho V \left(\begin{smallmatrix} k\\ \sigma\end{smallmatrix}\right) \left[\begin{smallmatrix} p_2\, k\\ \lambda_2\,\rho\end{smallmatrix}\right] \mathfrak{B}_1 \left[\begin{smallmatrix} p_1\, k\\ \lambda_1\,\rho\end{smallmatrix}\right],\end{split} \tag{28}$$

where $\mathfrak{B}_1 \left[\begin{smallmatrix} p_1\, p_2\\ \lambda_1\,\lambda_2\end{smallmatrix}\right] = \bar{v}(p_2,\lambda_2)\mathbf{M}_1 u(p_1,\lambda_1)$. The summation in the first two terms gets eliminated due to the diagonality property of U and V, see eq. (19), and leads to

$$\begin{split}&\mathcal{M}_1^{\rm ISR}\left(\begin{smallmatrix} p_1\, p_2\, k\\ \lambda_1\,\lambda_2\,\sigma\end{smallmatrix}\right) = \mathfrak{s}^{(1)}_\sigma(k)\mathfrak{B}_1 \left[\begin{smallmatrix} p_1\, p_2\\ \lambda_1\,\lambda_2\end{smallmatrix}\right] + r^{(1)} \left[\begin{smallmatrix} p_1\, p_2\, k\\ \lambda_1\,\lambda_2\,\sigma\end{smallmatrix}\right](k),\\ &r^{(1)} \left[\begin{smallmatrix} p_1\, p_2\, k\\ \lambda_1\,\lambda_2\,\sigma\end{smallmatrix}\right](k) = -\frac{eQ_e}{2kp_1} \sum_\rho \mathfrak{B}_1 \left[\begin{smallmatrix} k p_2\\ \rho \lambda_2\end{smallmatrix}\right] U \left(\begin{smallmatrix} k\\ \sigma\end{smallmatrix}\right) \left[\begin{smallmatrix} k p_1\\ \rho \lambda_1\end{smallmatrix}\right] + \frac{eQ_e}{2kp_2} \sum_\rho V \left(\begin{smallmatrix} k\\ \sigma\end{smallmatrix}\right) \left[\begin{smallmatrix} p_2\, k\\ \lambda_2\,\rho\end{smallmatrix}\right] \mathfrak{B}_1 \left[\begin{smallmatrix} p_1\, k\\ \lambda_1\,\rho\end{smallmatrix}\right],\\ &\mathfrak{s}^{(1)}_\sigma(k) = eQ_e\frac{b_\sigma(k,p_1)}{2kp_1} - eQ_e\frac{b_\sigma(k,p_2)}{2kp_2}, \quad |\mathfrak{s}^{(1)}_\sigma(k)|^2 = -\frac{e^2Q_e^2}{2}\left(\frac{p_1}{kp_1} - \frac{p_2}{kp_2}\right)^2.\end{split} \tag{29}$$

The soft part is now clearly separated and the remaining non-IR part, necessary for the CEEX, is obtained. The case of final state one real photon emission can be analysed in a similar way.

[7] It seems, however, that the γ-Z box for the heavy fermion is missing in the literature.

[8] This kind of separation was already exploited in refs. [19]. We thank E. Richter-Wąs for attracting our attention to this method.

6 First order 1-photon ISR+FSR

The first order, ISR+FSR, 1-photon matrix element, with explicit split into IR and non-IR parts, reads

$$\mathcal{M}_1^{(1)}\begin{pmatrix} p\,k \\ \lambda\sigma \end{pmatrix} = \mathfrak{s}_\sigma^{(1)}(k)\;\mathfrak{B}\begin{bmatrix} p \\ \lambda \end{bmatrix}(P-k) + \mathfrak{s}_\sigma^{(0)}(k)\;\mathfrak{B}\begin{bmatrix} p \\ \lambda \end{bmatrix}(P) + r^{(1)}\begin{bmatrix} p\,k \\ \lambda\sigma \end{bmatrix}(P-k) + r^{(0)}\begin{bmatrix} p\,k \\ \lambda\sigma \end{bmatrix}(P), \tag{30}$$

where we use the compact notation $\begin{bmatrix} p \\ \lambda \end{bmatrix} \equiv \begin{bmatrix} p_1\,p_2\,p_3\,p_4 \\ \lambda_1\,\lambda_2\,\lambda_3\,\lambda_4 \end{bmatrix}$, and the lowest order Born spin amplitudes $\mathfrak{B}$ are defined in eq. (23). The other ingredients are the initial state non-IR part:

$$r^{(1)}\begin{bmatrix} p\,k \\ \lambda\sigma \end{bmatrix}(X) = \frac{-eQ_e}{2kp_1}\sum_\rho U\begin{pmatrix} k \\ \sigma \end{pmatrix}\begin{bmatrix} p_1\,k \\ \lambda_1\,\rho \end{bmatrix}\mathfrak{B}\begin{bmatrix} k p_2\,p_3\,p_4 \\ \rho\lambda_2\,\lambda_3\,\lambda_4 \end{bmatrix}(X) + \frac{eQ_e}{2kp_2}\sum_\rho V\begin{pmatrix} k \\ \sigma \end{pmatrix}\begin{bmatrix} k p_2 \\ \rho\lambda_2 \end{bmatrix}\mathfrak{B}\begin{bmatrix} p_1\,k p_3\,p_4 \\ \lambda_1\,\rho\lambda_3\,\lambda_4 \end{bmatrix}(X) \tag{31}$$

and the final state non-IR part

$$r^{(0)}\begin{bmatrix} p\,k \\ \lambda\sigma \end{bmatrix}(X) = -\frac{eQ_f}{2kp_3}\sum_\rho U\begin{pmatrix} k \\ \sigma \end{pmatrix}\begin{bmatrix} k p_3 \\ \rho\lambda_3 \end{bmatrix}\mathfrak{B}\begin{bmatrix} p_1\,p_2\,k p_4 \\ \lambda_1\,\lambda_2\,\rho\lambda_4 \end{bmatrix}(X) + \frac{eQ_f}{2kp_4}\sum_\rho V\begin{pmatrix} k \\ \sigma \end{pmatrix}\begin{bmatrix} p_4\,k \\ \lambda_4\,\rho \end{bmatrix}\mathfrak{B}\begin{bmatrix} p_1\,p_2\,p_3\,k \\ \lambda_1\,\lambda_2\,\lambda_3\,\rho \end{bmatrix}(X). \tag{32}$$

The FSR $\mathfrak{s}$-factor

$$\mathfrak{s}_\sigma^{(0)}(k) = -eQ_f\frac{b_\sigma(k,p_3)}{2kp_3} + eQ_f\frac{b_\sigma(k,p_4)}{2kp_4}, \quad |\mathfrak{s}_\sigma^{(0)}(k)|^2 = -\frac{e^2Q_f^2}{2}\left(\frac{p_3}{kp_3} - \frac{p_4}{kp_4}\right)^2 \tag{33}$$

we define analogously to the ISR case.

7 Coherent exclusive exponentiation, zero and first order

Spin amplitudes in the zero-th order coherent exclusive exponentiation, $\mathcal{O}(\alpha^0)_{\rm CEEX}$, we define as follows

$$\mathfrak{M}_n^{(0)}\begin{pmatrix} p\,k_1\,k_2 & \dots & k_n \\ \lambda\sigma_1\,\sigma_2 & \dots & \sigma_n \end{pmatrix} = e^{\alpha B_4(p_1,\dots,p_4)}\sum_{\{\wp\}}\frac{X_\wp^2}{(p_3+p_4)^2}\mathfrak{B}\begin{bmatrix} p \\ \lambda \end{bmatrix}(X_\wp)\;\;\mathfrak{s}_{\sigma_1}^{\wp_1}(k_1)\mathfrak{s}_{\sigma_2}^{\wp_2}(k_2)\dots\mathfrak{s}_{\sigma_n}^{\wp_n}(k_n), \tag{34}$$

where the s-channel four-momentum in the resonance propagator is $X_\wp = p_1 + p_2 - \sum_{i=1}^n \wp_i\, k_i$. The partition $\wp$ is defined as a vector $(\wp_1, \wp_2, \dots, \wp_n)$ where $\wp_i = 1$ for ISR and $\wp_i = 0$ for FSR photon, see the analogous construction in refs. [5, 6]. For a given partition $X_\wp$ is therefore the total incoming four-momentum minus four-momenta of ISR photons. The *coherent* sum is taken over set $\{\wp\}$ of all 2^n partitions – this set is explicitly the following

$$\{\wp\} = \{(0,0,0,\dots,0),\ (1,0,0,\dots,0),\ (0,1,0,\dots,0),\ (1,1,0,\dots,0),\ \dots(1,1,1,\dots,1)\}. \tag{35}$$

In eq. (34) we profit from the Yennie-Frautschi-Suura [1] fundamental proof of factorization of all virtual IR corrections in the form-factor[9] $\exp(\alpha B_4)$, where

$$
\begin{aligned}
B_4(p_1,...,p_4) &= Q_e^2 B_2(p_1,p_2) + Q_f^2 B_2(p_3,p_4)\\
&\quad + Q_eQ_f B_2(p_1,p_3) + Q_eQ_f B_2(p_2,p_4) - Q_eQ_f B_2(p_1,p_4) - Q_eQ_f B_2(p_2,p_3).\\
B_2(p,q) &\equiv \int \frac{d^4k}{k^2-m_\gamma^2+i\epsilon}\,\frac{i}{(2\pi)^3}\left(\frac{2p+k}{k^2+2kp+i\epsilon}+\frac{2q-k}{k^2-2kq+i\epsilon}\right)^2.
\end{aligned}
\tag{36}
$$

In the above we assume that IR singularities are regularized with a finite photon mass m_γ which enters into all B_2's and implicitly into $\mathfrak{s}$-factors (and the real photon phase space integrals, in the following discussion).

The auxiliary factor $F = X^2/(p_3+p_4)^2$ is, from the formal point of view, not really necessary. Note that the F-factor does not affect the soft limit; it really matters if at least one very hard FSR photon is present. However, the F-factor is very useful, because it is present in the photon emission matrix element, both in $\mathcal{O}(\alpha^1)$ and also in all orders in the leading logarithmic (LL) approximation. It has also been present for a long time now in the "crude distribution" in the YFS-type Monte Carlo generators, see for instance ref. [3]. It is therefore natural to include it already in the $\mathcal{O}(\alpha^0)$ exponentiation. Otherwise, this F-factor will be included order by order. However, in such a case, the convergence of perturbative expansion will be deteriorated. As we shall see below, the introduction of the F-factor will slightly complicate the first order exponentiation.

The complete set of spin amplitudes for emission of n photons we define in $\mathcal{O}(\alpha^1)_{\rm CEEX}$ as follows:

$$
\begin{aligned}
&\mathcal{M}^{(1)}_n \left(\begin{smallmatrix} p\, k_1\, k_2 \\ \lambda\,\sigma_1\,\sigma_2\end{smallmatrix}\cdots\begin{smallmatrix} k_n \\ \sigma_n\end{smallmatrix}\right) = e^{\alpha B_4(p_1,\dots,p_4)}\\
&\quad\times \sum_{\{\wp\}}\prod_{i=1}^{n}\mathfrak{s}^{\wp_i}_{\sigma_i}(k_i)\left(\mathfrak{B}\left[\begin{smallmatrix} p\\ \lambda\end{smallmatrix}\right](X_\wp)\left(1+\delta^{(1)}_{Virt}\right)+\mathcal{R}_{\rm Box}\left[\begin{smallmatrix} p\\ \lambda\end{smallmatrix}\right](X_\wp)+\sum_{j=1}^{n}\mathcal{R}^{(\wp_j)}_1\left[\begin{smallmatrix} p\, k_j\\ \lambda\,\sigma_j\end{smallmatrix}\right](X_\wp)\right),\\
&\mathcal{R}^{(\omega)}_1\left[\begin{smallmatrix} p\, k\\ \lambda\,\sigma\end{smallmatrix}\right](X)\equiv\frac{1}{\mathfrak{s}^{\omega}_{\sigma}(k)}\left[r^{(\omega)}\left[\begin{smallmatrix} p\, k\\ \lambda\,\sigma\end{smallmatrix}\right](X)\;+\left(\frac{(p_3+p_4+\omega k_j)^2}{(p_3+p_4)^2}-1\right)\mathfrak{B}\left[\begin{smallmatrix} p\\ \lambda\end{smallmatrix}\right](X),\;\;\omega=\pm1\right].
\end{aligned}
\tag{37}
$$

The IR-finite $\delta^{(1)}_{Virt}$ and $\mathcal{R}_{\rm Box}$ are determined *unambiguously* by identifying for $n=0$ the above equation with eq. (25), up to terms of $\mathcal{O}(\alpha^1)$. We obtain

$$
\delta^{(1)}_{Virt}(s) = Q_e^2F_1(s,m_\gamma)+Q_f^2F_1(s,m_\gamma)-Q_e^2\alpha B_2(s,m_\gamma)-Q_f^2\alpha B_2(s,m_\gamma). \tag{38}
$$

The $\mathcal{R}_{\rm Box}$ is obtained from $\mathcal{M}_{\rm Box}$ by means of the substitution[10]

$$
f_{\rm BDP}(\bar{M}_B^2,m_\gamma,s,t,u)\to f_{\rm BDP}(\bar{M}_B^2,m_\gamma,s,t,u)-f_{\rm IR}(m_\gamma,t,u), \tag{39}
$$

[9] In the LL approximation it is, of course, the doubly-logarithmic Sudakov form-factor.

[10] In the above procedure of subtracting IR divergences, there is no reference to cut on photon energy, only reference to the photon mass, similar to the YFS exponentiation on squared spin-summed amplitudes.

where

$$f_{\rm IR}(m_\gamma,t,u) = \frac{2}{\pi}B_2(m_\gamma,t) - \frac{2}{\pi}B_2(m_\gamma,u) = \ln\left(\frac{t}{u}\right)\ln\left(\frac{m_\gamma^2}{\sqrt{tu}}\right) + \frac{1}{2}\ln\left(\frac{t}{u}\right). \tag{40}$$

Similarly the IR-finite $\mathcal{R}_1^{(\omega)}$ is determined *uniquely* by identifying, for $n=1$, eq. (37) with eq. (30). In particular the factor $F-1 = (p_3+p_4+\omega k_j)^2/(p_3+p_4)^2 - 1$ is a consequence of the introduction of the F-factor in eq. (34). If it was not included, then the 1-photon part in eq. (37) would not reduce to the amplitude of eq. (30). Thanks to the presence of $F-1$, for $n=1$, we recover in eq. (37) the correct first order amplitude of eq. (29).

For very narrow resonances the photon emission in the decay process is separated from the photon emission in the production process by very large time-space distance. The ISR*FSR interference is therefore strongly suppressed, typically by Γ/M factors. Since our real photons are present down to arbitrarily low $k^0_{\rm min} = \epsilon\sqrt{s}/2 \ll \Gamma$, the effects due to the resonance complex phase in the emission of the *real* photons are taken into account *numerically* and exactly. For *virtual* photons we have to sum up *analytically* certain subset of the ISR*FSR interferences to infinite order following Greco et al. [5,6]. In practice the rule is: multiply each part of the spin amplitude proportional to Z-propagator by the additional factor $\exp(\delta_G(s,t,u))$ where:

$$\delta_G(s,t,u) = -2Q_eQ_f\frac{\alpha}{\pi}\ln\left(\frac{t}{u}\right)\ln\left(\frac{M_Z^2 - iM_Z\Gamma_Z - s}{M_Z^2}\right) \tag{41}$$

In $\mathcal{O}(\alpha^1)$ the above exponential factor induces the additional subtraction in the γ-Z box: $\mathcal{M}_{\rm box}(s,t,u) \to \mathcal{M}_{\rm box}(s,t,u) - \delta_G(s,t,u)$. Strictly speaking the above improvement is not really necessary, because we would have obtained it order-by-order, through higher order virtual non-IR correction. In practice, however, it is mandatory. If we had not made it, then the ISR*FSR interference contribution to A_{FB} at Z peak from $\mathcal{O}(\alpha^1)_{\rm CEEX}$ would be dramatically wrong, i.e. 0.5% instead of 0.05%!

8 Differential cross sections and the YFS form-factor

The master formula for the unpolarized $\mathcal{O}(\alpha^r)_{\rm CEEX}$ total cross section is given by the standard quantum-mechanical expression of the type "matrix element squared modulus times phase space" (contrary to typical "parton shower" approach)

$$\sigma^{(r)} = \sum_{n=0}^{\infty}\frac{1}{n!}\int d\tau_n(p_1+p_2;\; p_3,p_4,\; k_1,\dots,k_n)\;\frac{1}{4}\sum_{\lambda,\sigma_1,\dots,\sigma_n=\pm}\left|\mathcal{M}_n^{(r)}\left({}^{p\,k_1\,k_2}_{\lambda\sigma_1\sigma_2}\cdots{}^{k_n}_{\sigma_n}\right)\right|^2, \tag{42}$$

where the Lorentz invariant phase space (LIPS) integration element is

$$\int d\tau_n(P;p_1,p_2,...p_n) \equiv \int (2\pi)^4\delta^{(4)}\left(P - \sum_{j=1}^n p_j\right)\prod_{j=1}^n\frac{d^3p_j}{2p_j^0(2\pi)^3}. \tag{43}$$

The above total cross section is perfectly IR-finite, as can be checked with a little bit of effort by *analytical* partial differentiation[11] with respect the photon mass

$$\frac{\partial}{\partial m_\gamma}\sigma^{(r)} = 0. \tag{44}$$

Furthermore, the integral of eq. (42) is perfectly implementable in the Monte Carlo form, using a method very similar to those in ref. [3]. Traditionally, however, the lower boundary on the real soft photons is defined using the energy cut condition $k^0 > \varepsilon\sqrt{s}/2$ in the laboratory frame. The practical advantage of such a cut is the lower photon multiplicity in the MC simulation, and consequently a faster computer program[12]. If the above energy cut on the photon energy is adopted, then the real soft-photon integral between the lower LIPS boundary defined by m_γ and that defined by ε can be evaluated by hand and summed up rigorously (the only approximation is $m_\gamma/m_e \to 0$) into an additional overall factor $\exp(2\alpha\tilde{B}_4(p_1,...,p_4))$, where

$$\begin{aligned}
\tilde{B}_4(p_1,...,p_4) &= Q_e^2\tilde{B}_2(p_1,p_2) + Q_f^2\tilde{B}_2(p_3,p_4)\\
&\quad + Q_eQ_f\tilde{B}_2(p_1,p_3) + Q_eQ_f\tilde{B}_2(p_2,p_4) - Q_eQ_f\tilde{B}_2(p_1,p_4) - Q_eQ_f\tilde{B}_2(p_2,p_3),\\
\tilde{B}_2(p,q) &\equiv \int\limits_{k^0<\varepsilon\sqrt{s}/2} \frac{d^3k}{k^0}\,\frac{(-1)}{8\pi^2}\left(\frac{p}{kp}-\frac{q}{kq}\right)^2.
\end{aligned} \tag{45}$$

Let us introduce $\mathfrak{M}_n^{(r)} = e^{-\alpha B_4}\mathcal{M}_n^{(r)}$ (without virtual IR singularities) and, altogether, the above reorganization yields the new expression for the unpolarized total cross-section

$$\sigma^{(r)} = \sum_{n=0}^{\infty}\frac{1}{n!}\int d\tau_n(p_1+p_2;\; p_3,p_4,\; k_1,\dots,k_n)\; e^{Y(p_1,\dots,p_4)}\;\frac{1}{4}\sum_{\lambda,\sigma_i=\pm}\left|\mathfrak{M}_n^{(r)}\left({}^{p\,k_1\,k_2}_{\lambda\sigma_1\sigma_2}\cdots{}^{k_n}_{\sigma_n}\right)\right|^2 \tag{46}$$

where $Y(p_1,...,p_4) = 2\alpha\tilde{B}_4(p_1,...,p_4) + 2\alpha\Re B_4(p_1,...,p_4)$ is the conventional YFS form-factor defined analytically in terms of logs and Spence functions – we do not show it here explicitly due to lack of space, see refs. [7, 11, 21, 22]. In the YFS form-factors we keep the final fermion mass exact. The fully exclusive differential cross section of eq. (46) is already implemented in the Monte Carlo event generator $\mathcal{KK}$ [11].

The extension of the above exponentiation procedure to $\mathcal{O}(\alpha^2)_{\rm CEEX}$ and beyond requires more work, but does not pose any conceptual problem. It will be implemented in the future version of the $\mathcal{KK}$ Monte Carlo.

[11] This method of validating IR-finiteness was noticed by G. Burgers [20]. The classical method of ref. [1] relies on the techniques of the Melin transform.

[12] The disadvantage of the cut $k^0 > \varepsilon\sqrt{s}/2$ is that in the MC it has to be implemented in *different* reference frames for ISR and for FSR – this costs the additional delicate procedure of bringing these two boundaries together, see ref. [11] and/or discussion in the analogous t-channel case in ref. [4].

9 Fermion spin polarization and photon spin randomization

The great advantage of working with spin amplitudes is the easiness of introduction of full spin polarizations for all particles. The general case of the total cross section with polarized beams and decays of unstable final fermion being sensitive to spin polarization [10, 23–25] reads

$$\sigma^{(r)} = \sum_{n=0}^{\infty} \frac{1}{n!} \int d\tau_n(p_1+p_2;\ p_3,p_4,\ k_1,\ldots,k_n)\ e^{Y(p_1,\ldots,p_4)} \sum_{\sigma_i} \sum_{a,b,c,d=0}^{3} \sum_{\lambda_i,\bar\lambda_i} \tag{47}$$
$$\hat\varepsilon_1^a \hat\varepsilon_2^b\ \sigma^a_{\lambda_1\bar\lambda_1} \sigma^b_{\lambda_2\bar\lambda_2} \mathfrak{M}^{(r)}_n \left({}^{p\,k_1\,k_2}_{\lambda\sigma_1\sigma_2}\cdots{}^{k_n}_{\sigma_n}\right) \left[\mathfrak{M}^{(r)}_n \left({}^{p\,k_1\,k_2}_{\bar\lambda\sigma_1\sigma_2}\cdots{}^{k_n}_{\sigma_n}\right)\right]^\star \sigma^c_{\bar\lambda_3\lambda_3} \sigma^d_{\bar\lambda_4\lambda_4} \hat h_3^c \hat h_4^d,$$

where, for $k = 1, 2, 3$, σ^k are Pauli matrices and $\sigma^0_{\lambda,\mu} = \delta_{\lambda,\mu}$ is the unit matrix. The components $\hat\varepsilon_1^a, \hat\varepsilon_2^b, a, b = 1, 2, 3$ are the components of the conventional spin polarization vectors of e^- and e^+ respectively, defined in the so-called GPS fermion rest frames (see ref [10] for the exact definition of these frames). We define $\hat\varepsilon_i^0 = 1$ in a non-standard way (i.e. $p_i \cdot \hat\varepsilon_i = m_e$). The *polarimeter* vectors $\hat h_i$ are similarly defined in the appropriate GPS rest frames of the final unstable fermions ($p_i \cdot \hat h_i = m_f$). Note that, in general, $\hat h_i$ may depend in a non-trivial way on momenta of all decay products, see refs. [24, 25] for details. We did not introduce polarimeter vectors for bremsstrahlung photons, i.e. we take advantage of the fact that the high energy experiment is completely blind to photon spin polarizations.

Let us finally touch briefly upon one very serious problem and its solution. In eq. (47) the single spin amplitude $\mathfrak{M}^{(1)}_n$ already contains $2^n(n+1)$ terms (due to 2^n ISR–FSR partitions). The grand sum over spins in eq. (47) counts $2^n 4^4 4^4 = 2^{n+16}$ terms! Altogether we expect up to $N \sim n2^{2n+16}$ operations in the CPU time expensive complex (16 bytes) arithmetics. Typically in $e^-e^+ \to \mu^-\mu^+$ the average photon multiplicity with $k^0 > 1$MeV is about 3, corresponding to $N \sim 10^7$ terms. In a sample of 10^4 MC events there will be a couple of events with $n = 10$ and $N = 10^{12}$ terms, clearly something that would "choke" completely any modern, fast workstation. There are several simple tricks that help to soften the problem; for instance, objects such as $\sum_a \hat\varepsilon_i^a \sigma^a_{\lambda\bar\lambda}$ and the $\mathfrak{s}$-factors are evaluated only once and stored for multiple use. This is however not sufficient. What really helps to substantially speed up the numerical calculation in the Monte Carlo program is the following trick of *photon spin randomization*. Instead of evaluating the sum over photon spins $\sigma_i,\ i = 1, ..., n$ in eq. (47), we generate randomly one spin sequence of $(\sigma_1, ..., \sigma_n)$ per MC event and the MC weight is calculated only for this particular spin sequence! In this way we save one hefty 2^n factor in the calculation time[13]. Mathematically this method is correct, i.e. the resulting cross section and all MC distribution will be the same as if we had used in the MC weight the original eq. (47) (see a formal proof of the above statement in Sect. 4 of ref. [26]). Let us stress again that it is possible to apply this photon spin randomization trick because (a) the typical high energy experiment is

[13]The other 2^n factor due to coherent summation over partitions cannot be eliminated, unless we give up on narrow resonances.

blind to photon spin polarization, so that we did not need to introduce in eq. (47) the polarimeter vectors for photons, and (b) For our choice of photon spin polarizations the cross section is rather weakly sensitive to them, so the method does not lead to significant loss in the MC efficiency.

10 Conclusions

We presented the first order coherent exclusive exponentiation CEEX scheme, with the full control over spin polarization for all fermions. This new method of exponentiation is very general and has many immediate and longer term advantages. The immediate profit will be the inclusion of the ISR–FSR interferences and availability of the exact distributions for multiple hard photons without giving up on exclusive, YFS-style, exponentiation. In particular it is applicable to diffucult case of the narrow resonances. The resulting spin amplitudes and the differential distributions are readily implemented in the MC event generator. (Numerical results will be presented elsewhere.)

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